

Growth equation with a conservation law

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A growth equation with a generalized conservation law characterized by an integral kernel is introduced. The equation contains the Kardar-Parisi-Zhang, Sun-Guo-Grant, and molecular-beam epitaxy growth equations as special cases and allows for a unified investigation of growth equations. From a dynamic renormalization-group analysis critical exponents and universality classes are determined for growth models with a conservation law.

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I. INTRODUCTION

In order to describe the dynamics and scaling of interface growth, Kardar, Parisi, and Zhang proposed a Langevin equation now known as the KPZ equation [1]. This equation is constructed to describe the long-time long-wavelength (hydrodynamic) limit of the dynamics of nonequilibrium interface growth processes. The KPZ equation has been studied intensively by analytical and numerical methods and a number of theoretical results have been obtained [2–5]. Specifically, the scaling of the correlation function $C(x,t) = x^{2\alpha} \bar{g}(t/x^2)$ characterized by the roughness exponent α and the dynamic exponent z has been established. However, only poor agreement is obtained between the theoretical results and experiments [4,5]. As a result, various modifications of the KPZ equation have been analyzed.

In the present paper we introduce a growth equation with a generalized conservation law described by an integral kernel. We will refer to the equation as the growth kernel equation (GKE). It contains the KPZ equation as a special case. In addition, the previously studied Sun-Guo-Grant (SGG) [6] and molecular-beam-epitaxy (MBE) [7] equations are also contained in our general equation.

The motivation for introducing the GKE equation is to gain information on how conservation laws change the universality classes for nonequilibrium growth models, and to allow for a *unified* description of growth models studied so far. Furthermore, one can speculate whether some growth experiments, which yield exponents that do not agree with the KPZ exponents, may contain nonlocal growth effects such as, e.g., the experiments on electrochemical deposition reported in Refs. [8,9].

Previous studies of nonlocal terms in interface related topics include fluctuating lines in quenched random environments. Domain walls subject to quenched long-range correlated impurities were studied in [10]. More recently, the dynamic relaxation of drifting polymers [11] and the critical dynamics of contact line depinning [12] were studied, where in both cases the equation of motion includes nonlocal interaction terms.

II. GROWTH KERNEL EQUATION

The GKE equation for a d dimensional interface $h(x,t)$ reads

$$\frac{\partial h}{\partial t} = \int d^d x' K(x-x') \left(\nu \nabla'^2 h + \frac{\lambda}{2} (\nabla' h)^2 \right) + \eta, \quad (1)$$

with an additive noise $\eta(x,t)$ whose correlations will be specified below. The kernel $K(r)$ describes nonlocal interactions in the system [13]. We want the total height $H(t) = \int d^d x h(x,t)$ to be conserved and impose the constraint $\int d^d x' K(x-x') = 0$, which leads to $\partial H / \partial t = 0$, provided the noise is chosen to satisfy $\eta(k=0,t) = 0$ [see Eq. (3) below]. Consequently, the GKE equation conserves the quantity $H(t)$.

The kernel has the behavior

$$K(r) \sim \frac{1}{r^{d+\sigma}} \text{ for } r \rightarrow \infty, \quad (2)$$

where we have introduced an exponent σ describing the long-distance decay. By Fourier transforming, $\sigma=0$ corresponds to the kernel being a Dirac δ function, and therefore the usual KPZ equation [1].

The case $K(x-x') = -\nabla_x^2 \delta^d(x-x')$ yields the dynamics of the SGG and MBE equations [6,7], and corresponds to $\sigma=2$. In order to incorporate these equations, we introduce a kernel $N(r)$ in the noise correlator

$$\langle \eta(x,t) \eta(x',t') \rangle = 2DN(x-x') \delta(t-t') \quad (3)$$

with the form

$$N(r) \sim \frac{1}{r^{d+\tau}} \text{ for } r \rightarrow \infty. \quad (4)$$

Here, τ is an exponent independent of σ . The case $\tau=0$ means no correlations in the noise (KPZ, MBE) whereas $\tau=2$ corresponds to conserved noise as it appears in, e.g., the SGG equation. The correlator (3) implies that $\eta(k=0,t) = 0$, as required above. We can imagine the noise related to a white noise $\xi(x,t)$ through $\eta(x,t) = \int d^d x' R(x-x') \xi(x',t)$, where the kernel $R(r)$ has the large argument behavior $R(r) \sim 1/r^{d+\tau/2}$. This form leads to the correlations in Eqs. (3) and (4).

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Under the rescaling $x \rightarrow x' = x/b$ the parameters in the GKE equation change as

$$\nu \rightarrow \nu' = b^{z-2-\sigma} \nu, \quad (5)$$

$$\lambda \rightarrow \lambda' = b^{z+\alpha-2-\sigma} \lambda, \quad (6)$$

$$D \rightarrow D' = b^{z-2\alpha-d-\tau} D. \quad (7)$$

For $\lambda=0$, the equation is made scale invariant for

$$z_0 = 2 + \sigma, \quad \alpha_0 = \frac{2 + \sigma - d - \tau}{2}. \quad (8)$$

If we use these values in the rescaling for λ we obtain that $\lambda' = b^{(2+\sigma-d-\tau)/2} \lambda$, so naively we expect the critical dimension of the model to be given by $d_c = 2 + \sigma - \tau$. Therefore, for $d > d_c$ the λ term will scale to zero, whereas for $d < d_c$ the λ term will be relevant and the scaling behavior of the GKE equation will no longer be described by the naive exponents. Now we will carry out a dynamic renormalization-group (RG) analysis in order to determine the scaling behavior of the GKE equation.

III. RENORMALIZATION-GROUP ANALYSIS

We Fourier transform the GKE equation using the rules for Fourier transformation of convolutions and products and obtain in the hydrodynamic limit $k \rightarrow 0$

$$\begin{aligned} h(k, \omega) = & G_0(k, \omega) \eta(k, \omega) - \frac{\lambda}{2} G_0(k, \omega) k^\sigma \int^\Lambda \frac{d^d q}{(2\pi)^d} \\ & \times \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} q \cdot (k-q) h(q, \Omega) h(k-q, \omega-\Omega), \end{aligned} \quad (9)$$

where $G_0(k, \omega)$ is the (bare) propagator defined by the expression $G_0(k, \omega) = 1/(\nu k^{2+\sigma} - i\omega)$. Λ is the momentum cutoff. The noise in Fourier space takes the form

$$\langle \eta(k, \omega) \eta(k', \omega') \rangle = 2Dk^\tau (2\pi)^{d+1} \delta^d(k+k') \delta(\omega+\omega'). \quad (10)$$

The renormalization group consists of coarse-graining followed by rescaling [14]. In coarse-graining, modes with momenta $e^{-l} < k < 1$ ($\Lambda \equiv 1$) are eliminated from the equation of motion. In rescaling, wave vectors are rescaled according to $k \rightarrow k' = bk$, with $b = e^l$. The RG procedure is most efficiently carried out by the means of diagrams, i.e., we represent the GKE equation (9) as shown in Fig. 1, cf. Refs. [1,15,16].

After a standard but lengthy calculation one obtains the one-loop RG flow for the GKE equation [17]

$$\frac{d\nu}{dl} = \nu \left(z - 2 - \sigma + K_d \frac{\lambda^2 D}{\nu^3} \frac{2 + 2\sigma - \tau - d}{4d} \right), \quad (11)$$

$$\frac{d\lambda}{dl} = \lambda(\alpha + z - 2 - \sigma), \quad (12)$$

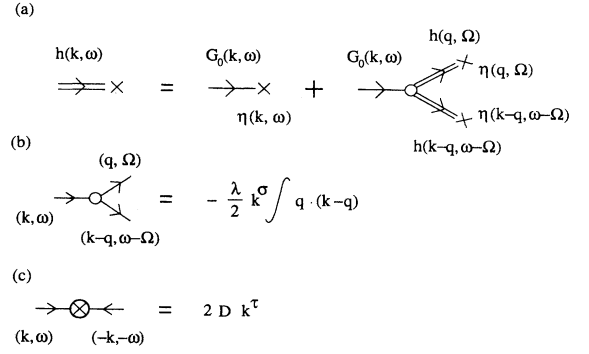


FIG. 1. (a) Diagrammatic representation of the GKE equation (9). (b) The vertex λ which includes integration over (q, Ω) . The $q \cdot (k-q)$ is associated with the outgoing momenta; $f \equiv \int^\Lambda [d^d q / (2\pi)^d] (d\Omega / 2\pi)$. (c) The contracted noise $2Dk^\tau$, from Eq. (10).

$$\frac{dD}{dl} = \begin{cases} D(z - 2\alpha - d - \tau), & \tau < 2\sigma, \\ D \left(z - 2\alpha - d - \tau + \frac{K_d \lambda^2 D}{4\nu^3} \right), & \tau \geq 2\sigma, \end{cases} \quad (13)$$

where $K_d = S_d / (2\pi)^d$, and $S_d = 2\pi^{d/2} / \Gamma(d/2)$ is the surface area of the d dimensional unit sphere. The nonrenormalization of the noise for $\tau < 2\sigma$ (a result which will be valid to all orders) leads to $z = 2\alpha + d + \tau$. Furthermore, the RG flow for the case $\tau \geq 2\sigma$ reduces to the case $\tau = 2\sigma$ [17]. Due to the fact that λ does not renormalize to one-loop order, cf. Eq. (12), one obtains the relation [18]

$$\alpha + z = 2 + \sigma. \quad (14)$$

The critical exponents are determined from $d\nu/dl = 0$ and $dD/dl = 0$, i.e., by fixing ν and D . It is convenient to introduce the coupling constant

$$g = g(l) = \frac{K_d \lambda^2 D}{4d \nu^3} \quad (15)$$

with the dimension $[L]^{d+\tau-\sigma-2}$. The RG flow of g for fixed ν and D becomes

$$\begin{aligned} \frac{dg}{dl} = & 2g \frac{1}{\lambda} \frac{d\lambda}{dl} = 2g(\alpha + z - \sigma - 2) \\ = & \begin{cases} (2 + \sigma - \tau - d)g - 3(2 + 2\sigma - \tau - d)g^2, & \tau < 2\sigma, \\ (2 + \sigma - \tau - d)g - [3(2 + 2\sigma - \tau) - 4d]g^2, & \tau \geq 2\sigma. \end{cases} \end{aligned} \quad (16)$$

The fixed points (FP's) for g are $g_0^* = 0$, and

$$g^* = \begin{cases} \frac{2 + \sigma - \tau - d}{3(2 + 2\sigma - \tau - d)}, & \tau < 2\sigma, \\ \frac{2 + \sigma - \tau - d}{3(2 + 2\sigma - \tau) - 4d}, & \tau \geq 2\sigma. \end{cases} \quad (17)$$

Physically, we have that $\nu, D > 0$. As a result, FP's where $g^* < 0$ are unphysical since they would lead to an imaginary value for λ .

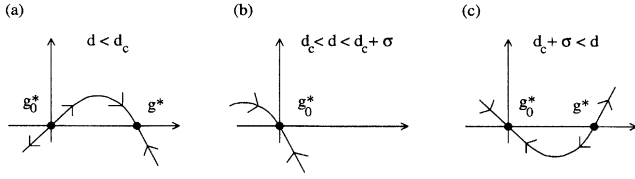


FIG. 2. Coupling constant flow for $\tau < 2\sigma$.

IV. RESULTS

First, we discuss the case $\tau < 2\sigma$. The critical dimension is $d_c = 2 + \sigma - \tau$. In Fig. 2 we show the RG flow of the coupling constant g , for various dimensions. For $d < d_c$ the trivial FP $g_0^* = 0$ is unstable, whereas g^* is stable. For $d > d_c$, g_0^* becomes stable. For $d > d_c + \sigma$, g^* is again positive but now an unstable FP. Probably this latter behavior is an artifact due to the fact that the FP is only known from the one-loop expansion [17].

The FP's and exponents can be calculated in an $\epsilon = d_c - d$ expansion. For the trivial FP $g_0^* = 0$ we recover the exponents for the linear GKE equation, cf. Eq. (8). Consequently, they will describe the GKE system for dimensions $d > d_c$, where the trivial FP is stable, cf. Fig. 2(b). The nontrivial FP is to first order in ϵ equal to $g^* = \epsilon/3\sigma$, with the exponents

$$\alpha = \frac{\epsilon}{3} = \frac{2 + \sigma - \tau - d}{3}, \quad (18)$$

$$z = 2 + \sigma - \frac{\epsilon}{3} = \frac{d + \tau + 2\sigma + 4}{3}. \quad (19)$$

These values are the exponents for the GKE equation in dimensions $d < d_c$, and are consistent with the nonrenormalization of λ and D in the case $\tau < 2\sigma$, cf. Eqs. (12) and (13).

Next, we discuss the case $\tau \geq 2\sigma$. We again remark that for any $\tau > 2\sigma$ we get the behavior for $\tau = 2\sigma$, and therefore we only have to discuss the latter case [17]. The critical dimension is $d_c = 2 - \sigma$. With $\epsilon = d_c - d$, the nontrivial FP becomes for $\sigma \neq \frac{1}{2}$

$$g^* = \frac{\epsilon}{2(2\epsilon + 2\sigma - 1)} = \frac{\epsilon}{2(2\sigma - 1)} + O(\epsilon^2). \quad (20)$$

The case $0 \leq \sigma \leq \frac{1}{2}$. Here, g^* is negative, and a FP expansion in powers of ϵ does not exist. The KPZ equation is a well-known example of this failure of the ϵ expansion. The two-loop results for the KPZ equation also show the failure

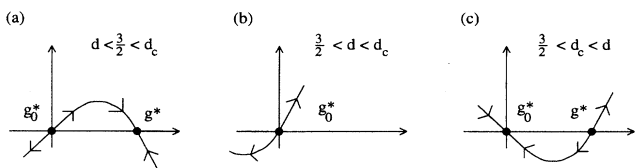


FIG. 3. Coupling constant flow for $\tau \geq 2\sigma$, and $\sigma \leq \frac{1}{2}$. For dimensions $d > d_c$ (c) shows the possibility of a phase transition at g^* (see the text).

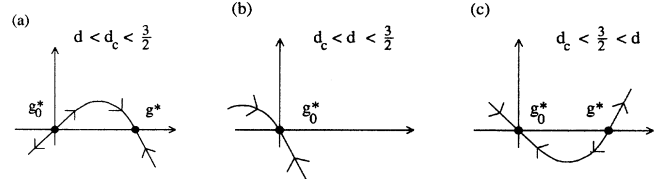


FIG. 4. Coupling constant flow for $\tau \geq 2\sigma$, and $\sigma > \frac{1}{2}$.

of the ϵ expansion around the critical dimension $d_c = 2$ [19]. In Fig. 3 we show the RG flow for g in the case where $\sigma < \frac{1}{2}$.

In order to obtain the exponents we can use the one-loop result (17) of the g^* fixed point. For the KPZ equation this gives the exact exponents in $d = 1$ (cf. [1]), but despite this “success” the method is uncontrolled due to the fact that g^* , or λ^* , is not small at the FP, which has been the underlying assumption under the whole RG calculation and series expansion in λ . The direct substitution of (17) into the expressions for the exponents results in the values (with $\tau = 2\sigma$)

$$\alpha = \frac{(2-d)(2-\sigma-d)}{2(3-2d)}, \quad (21)$$

$$z = 2 + \sigma - \frac{(2-d)(2-\sigma-d)}{2(3-2d)}, \quad (22)$$

which in $d = 1$ yields $\alpha = (1 - \sigma)/2$ and $z = 3(1 + \sigma)/2$. For $\sigma = 0$ this reduces to the KPZ exponents [1].

The case $\sigma > \frac{1}{2}$. In Fig. 4 we show the RG flow for g in this case. For dimensions $d > \frac{3}{2} > d_c$ the g^* fixed point is positive; probably this is an artifact due to the one-loop result. Now the ϵ expansion is possible. We can obtain the exponents to first order in ϵ at the $O(\epsilon)$ fixed point (20) with the result

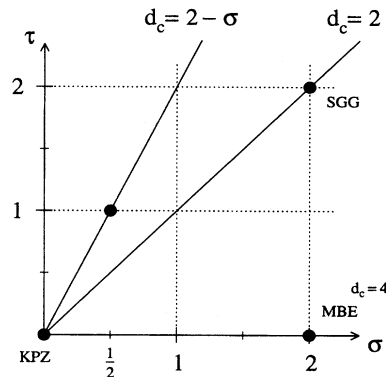


FIG. 5. Universality classes and critical dimensions for the GKE equation. Below the line $d_c = 2 - \sigma$, i.e., for $\tau < 2\sigma$, every point represents a distinct universality class. Above the line $d_c = 2 - \sigma$, i.e., for $\tau \geq 2\sigma$, every vertical line represents a different universality class. The KPZ, SGG, and MBE models are shown with solid circles. The circle at $\sigma = \frac{1}{2}, \tau = 1$ divides the $\tau = 2\sigma$ line into two parts. The part with $\sigma \leq \frac{1}{2}$ where the ϵ expansion does not exist, and the part $\sigma > \frac{1}{2}$ where the ϵ expansion does exist.

$$\alpha = \frac{\sigma}{2(2\sigma-1)} \epsilon + O(\epsilon^2), \quad (23)$$

$$z = 2 + \sigma - \frac{\sigma}{2(2\sigma-1)} \epsilon + O(\epsilon^2). \quad (24)$$

For $\tau = 2\sigma$ ($\tau \geq 2\sigma$) there is the possibility of carrying out an expansion in the quantity $\epsilon' = d - d_c = d + \sigma - 2$ for dimensions above d_c . The FP is

$$g^*(\epsilon') = \frac{1}{2(2\epsilon' + 1 - 2\sigma)} \epsilon' = \frac{1}{2(1 - 2\sigma)} \epsilon' + O(\epsilon'^2) \quad (25)$$

and it corresponds to a phase transition in the model, cf. Fig. 3(c). The phase transition is between the g_0^* “smooth” phase and the strong-coupling rough phase. When $\sigma > \frac{1}{2}$ the g^* fixed point is negative, i.e., the ϵ' expansion is only meaningful for $\sigma < \frac{1}{2}$. The exponents associated with the roughening phase transition are to first order in ϵ'

$$\alpha = \frac{\sigma}{2(1 - 2\sigma)} \epsilon', \quad z = 2. \quad (26)$$

For $\sigma = 0$ the exponents reduce to the well-known results $\alpha = 0$ and $z = 2$ for the KPZ equation (see, e.g., [19]).

V. CONCLUSIONS

In this paper we have performed a renormalization-group analysis of the GKE growth equation which is an equation with a generalized conservation law described by an integral

kernel. The results for the FP's and critical exponents have shown that the GKE equation encompasses a range of different universality classes; cf. Fig. 5.

For $\tau < 2\sigma$, every point represents a distinct universality class, with the SGG and MBE models belonging to this case. For all these universality classes we were able to obtain the critical exponents, and the values are given in Eqs. (18) and (19). Furthermore, the exponents fulfill the relation (14).

For $\tau \geq 2\sigma$, every vertical line represents a different universality class. The KPZ equation belongs to this case. Moreover, we noted the breakdown of the ϵ expansion for $\sigma < \frac{1}{2}$. As a consequence, estimates for the critical exponents could only be obtained by a direct substitution of the g^* FP value into the expressions for the exponents, resulting in the values in Eqs. (21) and (22). However, for $\sigma > \frac{1}{2}$ the ϵ expansion could be used to obtain the exponent values as given in Eqs. (23) and (24).

In the GKE equation we have the two free parameters τ and σ . As a result, we can always choose these values in order to obtain agreement with values for α and z determined in an experiment. However, unless one can argue that the experiment contains nonlocal growth effects described by a kernel this does not give the true explanation of the experiment.

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